数理解析研究所講究録
924 巻 1995 年 208-253

July 17, 1995

Split $\mathbb{Z}$-forms of irreducible prehomogeneous vector spaces

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Introduction.

Let G be a connected reductive group over $\mathbb{C}$, $\rho : G \to GL_n(\mathbb{C})$ a rational representation, and $V := \mathbb{C}^n$. Such a triple (G, ρ, V) is called a prehomogeneous vector space if G has a Zariski dense orbit in V. If (G, ρ, V) is an irreducible, (G, ρ, V) is said to be *irreducible.* Now assume that (G, ρ, V) is an irreducible prehomogeneous vector space such that there exist a non-trivial rational character $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ and an irreducible polynomial function $f \in \mathbb{C}[V]$ on V such that $f(gv) = \phi(g)f(v)$ for all $g \in G$ and $v \in V$. Put

$$\mathrm{Aut}(V, f) := \{(g, \phi_g) \in GL(V) \times \mathbb{C}^\times \mid f(gv) = \phi_g f(v) \text{ for all } v \in V\},$$

and $\mathrm{Aut}^0(V, f)$ be the identity component of $\mathrm{Aut}(V, f)$. If the image of $\mathrm{Aut}^0(V, f)$ by the first projection coincides with $\rho(G)$, then (G, ρ, V) is said to be *saturated.*

The purpose of this note is to classify and to describe the split $\mathbb{Z}$-forms of the saturated, irreducible prehomogeneous vector spaces. (See [**G**] for "split $\mathbb{Z}$-form".) For this purpose, we need to describe a Chevalley system explicitly for each complex simple Lie algebra. Such a description is given in §1, which would be useful in a different context, and so we have included some information which is not used in the present note. (For example, all information concerning E_8 is not necessary here.)

Notation. For a ring A ($\ni 1$), $M_n(A)$ denotes the totality of $n \times n$-matrices. The group of units in A is denoted by $A^\times$. An element of $A^\times$ is identified with the

the $n \times n$-matrix whose (i,j)-component is 1 and the other components are 0. We sometimes write E_i for E_{ii}. We denote by $\mathrm{diag}(t_1,\cdots,t_n)$ the diagonal matrix whose diagonal components are $t_1,\cdots,t_n$. For a set X, its cardinality is denoted by $\#X$.

§1. Chevalley system.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{C}$, $\mathfrak{h}$ a Cartan subalgebra, $\mathfrak{g} = \mathfrak{h} \oplus \sum_{r\in R}\mathfrak{g}(r)$ the root space decomposition, $0 \neq X(r) \in \mathfrak{g}(r)$, and $H(r)$ $(\in \mathfrak{h})$ the coroot vector which corresponds to a root r. A system $(X(r))_{r\in R}$ is called a *Chevalley system*, if

$$[X(r), X(-r)] = H(r) \qquad (r \in R)$$

and, for $r, s, r+s \in R$,

$$[X(r), X(s)] = \pm pX(r+s),$$

where p is the smallest positive integer such that $s + (p+1)r \notin R$.

The purpose of this section is to describe explicitly a Chevalley system for each complex simple Lie algebra.

1.1. Type A_{n-1}.

We may assume that

$$\mathfrak{g} = \{X \in M_n(\mathbf{C}) \mid \mathrm{tr}(x) = 0\}$$

and

$$\mathfrak{h} = \{\mathrm{diag}(t_1,\ldots,t_n) \mid t_i \in \mathbf{C}, \sum t_i = 0\}.$$

Then

$$R = \{\epsilon_i - \epsilon_j \mid i \neq j\},$$

where

$$\epsilon_i(\mathrm{diag}(t_1, \dots, t_n)) = t_i.$$

The coroots are given by

$$H(\epsilon_i - \epsilon_j) = E_i - E_j.$$

A Chevalley system is given by

$$X(\epsilon_i - \epsilon_j) = E_{ij}.$$

We may take as a root basis

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \qquad (1 \leq i \leq n-1).$$

Then the Dynkin diagram is given by

$$\underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\circ} \text{———} \cdots \text{———} \underset{\alpha_{n-1}}{\circ} .$$

1.2. Type B_n.

Let us define an element J of $M_{2n+1}(\mathbb{C})$ by

$$J = \sum_{i=1}^{n} (E_{i,n+i} + E_{n+i,i}) + 2E_{2n+1,2n+1}.$$

We may assume that

$$\mathfrak{g} = \{X \in M_{2n+1}(\mathbb{C}) \mid XJ + J\,{}^tX = 0\}$$

and

$$\mathfrak{h} = \{\mathrm{diag}(t_1, \dots, t_n, -t_1, \dots, -t_n, 0)\}.$$

Then

$$R = \{\pm\epsilon_i \pm \epsilon_j \quad (i \neq j), \pm\epsilon_i\},$$

where

$$\epsilon_i(\mathrm{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n, 0)) = t_i.$$

The coroots are given by

$$\begin{aligned}
&H(\epsilon_i - \epsilon_j) = (E_i - E_j) - (E_{n+i} - E_{n+j}) && (i \neq j)\\
&H(\epsilon_i + \epsilon_j) = (E_i + E_j) - (E_{n+i} + E_{n+j}) && (i < j)\\
&H(-\epsilon_i - \epsilon_j) = (-E_i - E_j) - (-E_{n+i} - E_{n+j}) && (i < j)\\
&H(\epsilon_i) = 2(E_i - E_{n+i})\\
&H(-\epsilon_i) = -2(E_i - E_{n+i}).
\end{aligned}$$

A Chevalley system is given by

$$\begin{aligned}
&X(\epsilon_i - \epsilon_j) = E_{ij} - E_{n+j,n+i} && (i \neq j)\\
&X(\epsilon_i + \epsilon_j) = E_{i,n+j} - E_{j,n+i} && (i < j)\\
&X(-\epsilon_i - \epsilon_j) = E_{n+j,i} - E_{n+i,j} && (i < j)\\
&X(\epsilon_i) = E_{i,2n+1} - 2E_{2n+1,n+i}\\
&X(-\epsilon_i) = 2E_{2n+1,i} - E_{n+i,2n+1}.
\end{aligned}$$

We may take as a root basis of R

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \qquad (1 \leq i < n), \qquad \alpha_n = \epsilon_n.$$

Then the Dynkin diagram is given by

$$\underset{\alpha_1}{\circ} \text{——} \underset{\alpha_2}{\circ} \text{——} \cdots \text{——} \underset{\alpha_{n-1}}{\circ} \Longrightarrow \underset{\alpha_n}{\circ}.$$

1.3. Type C_n.

Let

$$J = \sum_{i=1}^{n} (E_{i,n+i} - E_{n+i,i}).$$

We may assume that

$$\mathfrak{g} = \{X \in M_{2n}(\mathbb{C}) \mid XJ + J\,{}^tX = 0\}$$

and

$$\mathfrak{h} = \{\mathrm{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n)\}.$$

Then

$$R = \{\pm\epsilon_i \pm \epsilon_j \quad (i \neq j), \quad \pm 2\epsilon_i\},$$

where

$$\epsilon_i(\mathrm{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n)) = t_i.$$

The coroots are given by

$$\begin{aligned}
&H(\epsilon_i - \epsilon_j) = (E_i - E_j) - (E_{n+i} - E_{n+j}) && (i \neq j)\\
&H(\epsilon_i + \epsilon_j) = (E_i + E_j) - (E_{n+i} + E_{n+j}) && (i < j)\\
&H(-\epsilon_i - \epsilon_j) = -(E_i + E_j) + (E_{n+i} + E_{n+j}) && (i < j)\\
&H(2\epsilon_i) = E_i - E_{n+i}\\
&H(-2\epsilon_i) = -E_i + E_{n+i}.
\end{aligned}$$

A Chevalley system is given by

$$\begin{aligned}
&X(\epsilon_i - \epsilon_j) = E_{i,j} - E_{n+j,n+i} && (i \neq j)\\
&X(\epsilon_i + \epsilon_j) = E_{i,n+j} + E_{j,n+i} && (i < j)\\
&X(-\epsilon_i - \epsilon_j) = E_{n+j,i} + E_{n+i,j} && (i < j)\\
&X(2\epsilon_i) = E_{i,n+i}\\
&X(-2\epsilon_i) = E_{n+i,i}.
\end{aligned}$$

We may take as a root basis

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i < n), \qquad \alpha_n = 2\epsilon_n.$$

Then the Dynkin diagram is given by

$$\underset{\alpha_1}{\circ} \text{---} \underset{\alpha_2}{\circ} \text{---} \cdots \text{---} \underset{\alpha_{n-1}}{\circ} \Longleftarrow \underset{\alpha_n}{\circ}.$$

1.4. Type D_n.

Let

$$J = \sum_{i=1}^{n} (E_{i,n+i} + E_{n+i,i}).$$

We may assume that

$$\mathfrak{g} = \{X \in M_{2n}(\mathbb{C}) \mid XJ + J\,{}^tX = 0\}$$

and

$$\mathfrak{h} = \{\mathrm{diag}(t_1, \dots, t_n, -t_1, \dots, -t_n)\}.$$

Then

$$R = \{\pm\epsilon_i \pm \epsilon_j \quad (i \neq j)\},$$

where

$$\epsilon_i(\mathrm{diag}(t_1, \dots, t_n - t_n, \dots, -t_n)) = t_i.$$

The coroots are given by

$$\begin{aligned}
H(\epsilon_i - \epsilon_j) &= (E_i - E_j) - (E_{n+i} - E_{n+j}) && (i \neq j) \\
H(\epsilon_i + \epsilon_j) &= (E_i + E_j) - (E_{n+i} + E_{n+j}) && (i < j) \\
H(-\epsilon_i - \epsilon_j) &= -(E_i + E_j) + (E_{n+i} + E_{n+j}) && (i < j).
\end{aligned}$$

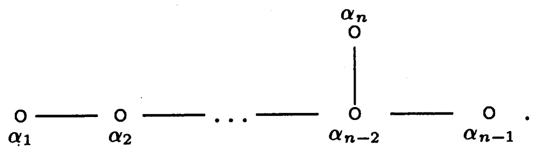
A Chevalley system is given by

$$
\begin{aligned}
X(\epsilon_i - \epsilon_j) &= E_{ij} - E_{n+j,n+i} && (i \neq j) \\
X(\epsilon_i + \epsilon_j) &= E_{i,n+j} - E_{j,n+i} && (i < j) \\
X(-\epsilon_i - \epsilon_j) &= E_{n+j,i} - E_{n+i,j} && (i < j).
\end{aligned}
$$

We may take as a root basis

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i < n), \quad \alpha_n = \epsilon_{n-1} + \epsilon_n.$$

Then the Dynkin diagram is given by



Up to now, we have worked with the vector representation of the simple Lie algebra of type D_n, but we also need to work with the half-spin representation. In the remainder of this paragraph, we freely use the notations of [**SK**,pp.110-114], where a brief account of the theory of the spin representation is given.

The representation space $\Lambda(E) = \Lambda(\mathbf{C}^n)$ of the spin representation is the Grassmann algebra of the vector space $E = \oplus_{i=1}^n \mathbf{C} e_i$. We write $e_{i_1} e_{i_2} \dots e_{i_k}$ for $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$. Let us consider two kinds of linear operators which are defined as follows:

$$
\begin{aligned}
e_i(e_{i_1} e_{i_2} \dots e_{i_k}) &= e_i e_{i_1} e_{i_2} \dots e_{i_k}. \\
f_i(e_{i_1} e_{i_2} \dots e_{i_k}) &= \begin{cases} (-1)^{p-1} e_{i_1} \dots \hat{e_{i_p}} \dots e_{i_k}, & \text{if } i = i_p \text{ for some } p, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Here $e_{i_1} \dots \hat{e_{i_p}} \dots e_{i_k}$ means $e_{i_1} \dots e_{i_{p-1}} e_{i_{p+1}} \dots e_{i_k}$. Let $\tilde{\mathfrak{g}}$ be the linear span of

$$\begin{array}{ll} e_i f_j & (1 \leq i, j \leq n), \\ e_i e_j & (1 \leq i < j \leq n), \\ f_j f_i & (1 \leq i < j \leq n). \end{array}$$

Then $\tilde{\mathfrak{g}}$ is a Lie algebra and an isomorphism between $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ is given as follows:

$$\begin{array}{lcll} \mathrm{diag}(t_1, \dots, t_n - t_1, \dots, -t_n) & \leftrightarrow & \frac{1}{2}\sum_{i=1}^{n} t_i(e_i f_i - f_i e_i). & \\ X(\epsilon_i - \epsilon_j) = E_{ij} - E_{n+j,n+i} & \leftrightarrow & e_i f_j & (i \neq j), \\ X(\epsilon_i + \epsilon_j) = E_{i,n+j} - E_{j,n+i} & \leftrightarrow & e_i e_j & (i < j), \\ X(-\epsilon_i - \epsilon_j) = E_{n+j,i} - E_{n+i,j} & \leftrightarrow & f_j f_i & (i < j). \end{array}$$

Thus a Chevalley system of $\tilde{\mathfrak{g}}$ is given by

$$\begin{array}{ll} X(\epsilon_i - \epsilon_j) = e_i f_j & (i \neq j), \\ X(\epsilon_i + \epsilon_j) = e_i e_j & (i < j), \\ X(-\epsilon_i - \epsilon_j) = f_j f_i & (i < j). \end{array}$$

As is easily seen

$$\Lambda^{odd} = \Lambda^{odd}(E) = \sum_{k=odd} \Lambda^k(E)$$

and

$$\Lambda^{even} = \Lambda^{even}(E) = \sum_{k=even} \Lambda^k(E)$$

are $\tilde{\mathfrak{g}}$-stable subspaces of $\Lambda(E)$. These $\tilde{\mathfrak{g}}$-modules Λ^{odd} and Λ^{even} are known to be irreducible and are called the *odd half-spin representation* and the *even half-spin representation*, respectively.

We define an involutory automorphism ι of the Clifford algebra $C(Q)$ (generated by $\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$) by $\iota(e_i)=f_i$ and $\iota(f_i)=e_i$ $(1\le i\le n)$. Then ι induces an automorphism of $Spin_{2n}$, which we shall denote by the same letter ι. See [**SK**, pp.110–114] for the Clifford algebras and the spin groups.

1.5. Type G_2.

We may assume that $\mathfrak{g}$ is the totality of the matrixes

$$\begin{pmatrix} 0 & 2d & 2e & 2f & 2a & 2b & 2c \\ a & x_{11} & x_{12} & x_{13} & 0 & f & -e \\ b & x_{21} & x_{22} & x_{23} & -f & 0 & d \\ c & x_{31} & x_{32} & x_{33} & e & -d & 0 \\ d & 0 & -c & b & -x_{11} & -x_{21} & -x_{31} \\ e & c & 0 & -a & -x_{12} & -x_{22} & -x_{32} \\ f & -b & a & 0 & -x_{13} & -x_{23} & -x_{33} \end{pmatrix}$$

with $x_{11}+x_{22}+x_{33}=0$, and

$$\mathfrak{h}=\{\mathrm{diag}(0,t_1,t_2,t_3,-t_1,-t_2,-t_3)\mid t_1+t_2+t_3=0\}.$$

Then

$$R=\{\epsilon_i-\epsilon_j\quad(i\neq j),\quad \pm\epsilon_i\},$$

where

$$\epsilon_i(\mathrm{diag}(0,t_1,t_2,t_3,-t_1,-t_2,-t_3))=t_i.$$

The coroots are given by

$$H(\epsilon_i-\epsilon_j)=(E_{1+i}-E_{1+j})-(E_{4+i}-E_{4+j})\qquad(i\neq j),$$

$$H(\epsilon_i)=(2E_{1+i}-E_{1+j}-E_{1+k})-(2E_{4+i}-E_{4+j}-E_{4+k}),$$

$$H(-\epsilon_i)=-(2E_{1+i}-E_{1+j}-E_{1+k})+(2E_{4+i}-E_{4+j}-E_{4+k}),$$

where $\{i,j,k\} = \{1,2,3\}$. A Chevalley system is given by

$$X(\epsilon_i - \epsilon_j) = E_{1+i,1+j} - E_{4+j,4+i},$$

$$X(\epsilon_i) = E_{1+i,1} + 2E_{1,4+i} + E_{4+k,1+j} - E_{4+j,1+k},$$

$$X(-\epsilon_i) = E_{4+i,1} + 2E_{1,1+i} + E_{1+j,4+k} - E_{1+k,4+j},$$

where (i,j,k) is an arbitrary even permutation of $(1,2,3)$. In fact,

$$[X(\epsilon_i - \epsilon_j), X(\epsilon_j - \epsilon_k)] = X(\epsilon_i - \epsilon_k)$$

$$[X(\epsilon_i - \epsilon_j), X(\epsilon_j)] = X(\epsilon_i)$$

$$[X(\epsilon_i - \epsilon_j), X(-\epsilon_i)] = -X(-\epsilon_j)$$

$$[X(\epsilon_i), X(-\epsilon_j)] = 3X(\epsilon_i - \epsilon_j),$$

$$[X(\epsilon_i), X(\epsilon_j)] = 2X(-\epsilon_k),$$

$$[X(-\epsilon_i), X(-\epsilon_j)] = -2X(\epsilon_k).$$

In the last two commutation relations, $\{i,j,k\} = \{1,2,3\}$. Let $\mathfrak{C}$ be the octonion algebra (=the algebra of Cayley numbers) over $\mathbf{C}$ [**F**,1.1]. Define a basis of $\mathfrak{C}$ by

$$\begin{array}{lll} u_1 = e_0, & u_2 = e_7 & \\ u_3 = e_1 + \sqrt{-1}e_6, & u_4 = e_2 + \sqrt{-1}e_5, & u_5 = e_4 + \sqrt{-1}e_3, \\ u_6 = -e_1 + \sqrt{-1}e_6, & u_7 = -e_2 + \sqrt{-1}e_5, & u_8 = -e_4 + \sqrt{-1}e_3. \end{array}$$

Here we use the notations of [**F**,1.5]. With respect to this basis, the Lie algebra of the infinitesimal automorphisms of $\mathfrak{C}$ is identified with the Lie algebra $\mathfrak{g}$ defined above. We may take as a root basis

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = -\epsilon_1.$$

Then the Dynkin diagram is given by

$$\alpha_1 \Rrightarrow \alpha_2.$$

1.6. Type F_4.

In this paragraph, we use the notations of [**F**]. Define a basis of $\mathfrak{C}$ by

$$(1.6.1)\qquad \begin{aligned} f_1 &= e_0 + \sqrt{-1}e_7, & f_5 &= -e_0 + \sqrt{-1}e_7, \\ f_2 &= e_6 + \sqrt{-1}e_1, & f_6 &= -e_6 + \sqrt{-1}e_1, \\ f_3 &= e_5 + \sqrt{-1}e_2, & f_7 &= -e_5 + \sqrt{-1}e_2, \\ f_4 &= e_3 + \sqrt{-1}e_4, & f_8 &= -e_3 + \sqrt{-1}e_4. \end{aligned}$$

The multiplication table is given by

(1.6.2)

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1	$2f_1$	$2f_2$	$2f_3$	$2f_4$	0	0	0	0
f_2	0	0	$-2f_8$	$2f_7$	$-2f_2$	$2f_1$	0	0
f_3	0	$-2f_8$	0	$-2f_6$	$-2f_3$	0	$2f_1$	0
f_4	0	$-2f_7$	$2f_6$	0	$-2f_4$	0	0	$2f_1$
f_5	0	0	0	0	$-2f_5$	$-2f_6$	$-2f_7$	$-2f_8$
f_6	$2f_6$	$-2f_5$	0	0	0	0	$2f_4$	$-2f_3$
f_7	$2f_7$	0	$-2f_5$	0	0	$-2f_4$	0	$2f_2$
f_8	$2f_8$	0	0	$-2f_5$	0	$2f_3$	$-2f_2$	0

e.g., $f_1f_3 = 2f_3$, $f_3f_1 = 0$. Let us identify a linear endomorphism of $\mathfrak{C}$ with the corresponding matrix with respect to the basis $\{f_i\}$, e.g., $E_{ij}f_j = f_i$. Let us describe the automorphisms λ and λ^2 of $\mathfrak{D}_4$ [**F**,2.2.4] in the matrix form. For

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & 0 & y_{21} & y_{31} & y_{41} \\ x_{21} & 0 & x_{23} & x_{24} & -y_{21} & 0 & y_{32} & y_{42} \\ x_{31} & x_{32} & 0 & x_{34} & -y_{31} & -y_{32} & 0 & y_{43} \\ x_{41} & x_{42} & x_{43} & 0 & -y_{41} & -y_{42} & -y_{43} & 0 \\ 0 & -z_{12} & -z_{13} & -z_{14} & 0 & -x_{21} & -x_{31} & -x_{41} \\ z_{12} & 0 & -z_{23} & -z_{24} & -x_{12} & 0 & -x_{32} & -x_{42} \\ z_{13} & z_{23} & 0 & -z_{34} & -x_{13} & -x_{23} & 0 & -x_{43} \\ z_{14} & z_{24} & z_{34} & 0 & -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix},$$

we have

$$\lambda(X) = \begin{pmatrix} 0 & -y_{43} & y_{42} & -y_{32} & 0 & -x_{21} & -x_{31} & -x_{41} \\ -z_{34} & 0 & x_{23} & x_{24} & x_{21} & 0 & z_{14} & -z_{13} \\ z_{24} & x_{32} & 0 & x_{34} & x_{31} & -z_{14} & 0 & z_{12} \\ -z_{23} & x_{42} & x_{43} & 0 & x_{41} & z_{13} & -z_{12} & 0 \\ 0 & x_{12} & x_{13} & x_{14} & 0 & z_{34} & -z_{24} & z_{23} \\ -x_{12} & 0 & -y_{41} & y_{31} & y_{43} & 0 & -x_{32} & -x_{42} \\ -x_{13} & y_{41} & 0 & -y_{21} & -y_{42} & -x_{23} & 0 & -x_{43} \\ -x_{14} & -y_{31} & y_{21} & 0 & y_{32} & -x_{24} & -x_{34} & 0 \end{pmatrix} \tag{1.6.3}$$

and

$$\lambda^2(X) = \begin{pmatrix} 0 & -z_{12} & -z_{13} & -z_{14} & 0 & z_{34} & -z_{24} & z_{23} \\ -y_{21} & 0 & x_{23} & x_{24} & -z_{34} & 0 & -x_{14} & x_{13} \\ -y_{31} & x_{32} & 0 & x_{34} & z_{24} & x_{14} & 0 & -x_{12} \\ -y_{41} & x_{42} & x_{43} & 0 & -z_{23} & -x_{13} & x_{12} & 0 \\ 0 & -y_{43} & y_{42} & -y_{32} & 0 & y_{21} & y_{31} & y_{41} \\ y_{43} & 0 & x_{41} & -x_{31} & z_{12} & 0 & -x_{32} & -x_{42} \\ -y_{42} & -x_{41} & 0 & x_{21} & z_{13} & -x_{23} & 0 & -x_{43} \\ y_{32} & x_{31} & -x_{21} & 0 & z_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix}. \tag{1.6.4}$$

Let

$$\begin{pmatrix} t_1^{(0)} \\ t_2^{(0)} \\ t_3^{(0)} \\ t_4^{(0)} \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}, \quad \begin{pmatrix} t_1^{(j+1)} \\ t_2^{(j+1)} \\ t_3^{(j+1)} \\ t_4^{(j+1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} t_1^{(j)} \\ t_2^{(j)} \\ t_3^{(j)} \\ t_4^{(j)} \end{pmatrix}.$$

Then, for

$$X = \mathrm{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4),$$

we have

$$\lambda^j(X) = \mathrm{diag}(t_1^{(j)}, t_2^{(j)}, t_3^{(j)}, t_4^{(j)}, -t_1^{(j)}, -t_2^{(j)}, -t_3^{(j)}, -t_4^{(j)}). \tag{1.6.5}$$

As in [**F**,4.5.9], $\mathfrak{J}$ denotes the exceptional simple Jordan algebra. We may assume that

$$\mathfrak{g} = \{\text{infinitesimal automorphisms of } \mathfrak{J}\}.$$

Let us identify an element δ of $\mathfrak{D}_4$ with the element δ of $\mathfrak{g}$ defined by

$$\delta \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & \delta_3 x_3 & \overline{\delta_2 x_2} \\ \overline{\delta_3 x_3} & 0 & \delta_1 x_1 \\ \delta_2 x_2 & \overline{\delta_1 x_1} & 0 \end{pmatrix},$$

where $\delta_i = \lambda^{i-1}(\delta)$. We may assume that

$$\mathfrak{h} = \{\mathrm{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4)\},$$

where we identify $\mathfrak{h}$ $(\subset \mathfrak{D}_4)$ with a subalgebra of $\mathfrak{g}$ via the above defined identification. Then

$$R = \{\pm\epsilon_i \pm \epsilon_j \ (i \neq j),\ \pm\epsilon_i,\ \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\},$$

where

$$\epsilon_i(\mathrm{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4)) = t_i.$$

The coroots are given by

$$\begin{aligned}
&H(s_i\epsilon_i + s_j\epsilon_j) = s_i(E_i - E_{4+i}) + s_j(E_j - E_{4+j}) \quad (i \neq j) \\
(1.6.6) \qquad &H(s_i\epsilon_i) = s_i(2E_i - 2E_{4+i}) \\
&H(\frac{1}{2}(s_1\epsilon_1 + s_2\epsilon_2 + s_3\epsilon_3 + s_4\epsilon_4)) = \sum_{i=1}^{4} s_i(E_i - E_{4+i}),
\end{aligned}$$

where $s_i = \pm 1$. For $a \in \mathfrak{C}$, let

$$(a)_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix}, (a)_2 = \begin{pmatrix} 0 & 0 & -\overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, (a)_3 = \begin{pmatrix} 0 & a & 0 \\ -\overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $X \in \mathfrak{M}^{(3)}$, define a linear endomorphism $\tilde{X}$ of $\mathfrak{J}$ by

$$\tilde{X}(Y) = \frac{1}{2}(XY + Y^*X^*),$$

where X^* is the transposed conjugate of X [**F**,4.1]. A Chevalley system is given by

$$\begin{aligned}
&X(\epsilon_i - \epsilon_j) = E_{ij} - E_{4+j,4+i} \qquad (i \neq j) \\
&X(\epsilon_i + \epsilon_j) = E_{i,4+j} - E_{j,4+i} \qquad (i < j) \\
&X(-\epsilon_i - \epsilon_j) = E_{4+j,i} - E_{4+i,j} \quad (i < j) \\
(1.6.7) \qquad &X(\epsilon_i) = (f_i)_1^{\sim} \qquad\qquad X(-\epsilon_i) = (f_{4+i})_1^{\sim} \\
&X(\epsilon_i \circ \lambda) = (f_i)_2^{\sim} \qquad\qquad X(-\epsilon \circ \lambda) = (f_{4+i})_2^{\sim} \\
&X(\epsilon_i \circ \lambda^2) = (f_i)_3^{\sim} \\
&X(-\epsilon \circ \lambda^2) = (f_{4+i})_3^{\sim}.
\end{aligned}$$

Note that

$$\begin{pmatrix} \epsilon_1 \circ \lambda \\ \epsilon_2 \circ \lambda \\ \epsilon_3 \circ \lambda \\ \epsilon_4 \circ \lambda \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix}$$

and

$$\begin{pmatrix} \epsilon_1 \circ \lambda^2 \\ \epsilon_2 \circ \lambda^2 \\ \epsilon_3 \circ \lambda^2 \\ \epsilon_4 \circ \lambda^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix}.$$

Let us give explicitly the commutation relations. Let δ be an element of $\mathfrak{g}$, of the form $X(\pm\epsilon_i \pm \epsilon_j)$ $(i \neq j)$. Then $\delta_1, \delta_2, \delta_3$ are of the form $\pm X(\pm\epsilon_i \pm \epsilon_j)$ by (1.6.3) and (1.6.4). Here $\delta_j f_i$ are of the form $\pm f_k$. By [**F**,4.9.4],

$$[\delta, (f_i)_j^\sim] = (\delta_j f_i)_j^\sim = \pm (f_k)_j^\sim \text{ or } 0. \tag{1.6.8}$$

The signature appeared in (1.6.8) can be easily determined by using (1.6.3), (1.6.4) and (1.6.7). A direct calculation shows that

$$\begin{aligned} [(f_i)_1^\sim, (f_j)_1^\sim] &= 2(E_{ij'} - E_{ji'}) \\ [(f_i)_2^\sim, (f_j)_2^\sim] &= 2\lambda^2(E_{ij'} - E_{ji'}) \\ [(f_i)_3^\sim, (f_j)_3^\sim] &= 2\lambda(E_{ij'} - E_{ji'}), \end{aligned} \tag{1.6.9}$$

where

$$i' = \begin{cases} i+4, & (i \leq 4) \\ i-4, & (i > 4), \end{cases}$$

and

$$[(a)_i^\sim, (b)_j^\sim] = (-\frac{1}{2}\overline{ab})_k^\sim, \tag{1.6.10}$$

for each even permutation (i, j, k) of $(1,2,3)$. Since $-\frac{1}{2}\overline{f_i f_j}$ is of the form $\pm f_k$ of 0, (1.6.8), (1.6.9) and (1.6.10) together with the results of (1.4), give the commutation relation among the Chevalley system given above. We may take as a root basis

$$\begin{aligned}
&\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4,\\
&\alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).
\end{aligned}$$

Then the Dynkin diagram is given by

$$\underset{\alpha_1}{\circ} \text{---} \underset{\alpha_2}{\circ} \Longrightarrow \underset{\alpha_3}{\circ} \text{---} \underset{\alpha_4}{\circ}\ .$$

1.7. Type E_6.

In this paragraph, we use the notations of [**F**]. We may assume that

$$\begin{aligned}
\mathfrak{g} = \mathfrak{E}_6 = \{&\text{linear endomorphisms of } \mathfrak{J} \text{ which (infinitesimally)}\\
&\qquad\qquad\text{preserves } \det(X, Y, Z)\}
\end{aligned}$$

[**F**,8.1]. The Lie algebra $\mathfrak{F}_4$ of infinitesimal automorphisms of $\mathfrak{J}$ is contained in $\mathfrak{g}$. Let $\mathfrak{h}_4$ be the Cartan subalgebra of $\mathfrak{F}_4$ which is given in (1.6). We may assume that

$$\mathfrak{h} = \mathfrak{h}_4 + \{\begin{pmatrix} t_5 & & \\ & t_6 & \\ & & t_7 \end{pmatrix}^{\sim} \mid t_5 + t_6 + t_7 = 0\}.$$

Let

$$h(t_1, \ldots, t_7) = \mathrm{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4) + \begin{pmatrix} t_5 & & \\ & t_6 & \\ & & t_7 \end{pmatrix}^{\sim}$$

and

$$\epsilon_i(h(t_1,\ldots,t_7)) = t_i.$$

Let us define endomorphisms α_{ij} $(1 \le i,j \le 3)$ of $\mathfrak{D}_4$ by

$$\alpha_{ii} = 0 \quad (1 \le i \le 3),$$

$$\alpha_{23} = 1, \qquad \alpha_{31} = \lambda, \qquad \alpha_{12} = \lambda^2,$$

$$\alpha_{32} = \kappa, \qquad \alpha_{13} = \kappa\lambda, \qquad \alpha_{21} = \kappa\lambda^2$$

[**F**,2.2]. Note that every α_{ij} preserves $\mathfrak{h}_4$. Let

$$A_{ii} = 0 \quad (1 \le i \le 3),$$

$$A_{23} = 1,$$

$$A_{31} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

$$A_{12} = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$A_{32} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_{13} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

$$A_{21} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} t_1^{ij} \\ t_2^{ij} \\ t_3^{ij} \\ t_4^{ij} \end{pmatrix} = A_{ij}\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

Then

$$\alpha_{ij}h(t_1, t_2, t_3, t_4, 0, 0, 0) = h(t_1^{ij}, t_2^{ij}, t_3^{ij}, t_4^{ij}, 0, 0, 0).$$

We identify an element δ of $\mathfrak{D}_4$ with a linear endomorphism of $\mathfrak{M}_3$ [**F**,4.1] as follows:

$$\delta(\sum_{i,j=1}^{3} x_{ij}E_{ij}^{(3)}) - \sum_{i,j=1}^{3}(\delta_{ij}x_{ij})E_{ij}^{(3)},$$

where $\delta_{ij} = \alpha_{ij}(\delta)$ [**F**,4.9]. Let

$$R_{ij} = \{\pm\epsilon_k \circ \alpha_{ij} \mid 1 \leq k \leq 4\} \quad (1 \leq i, j \leq 3).$$

Then

$$\begin{aligned} R &= \bigcup_{i\neq j}(R_{ij} + \frac{1}{2}(\epsilon_{4+i} - \epsilon_{4+j})) \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 4\} \\ &= \{\pm\epsilon_i \pm \frac{1}{2}(\epsilon_6 - \epsilon_7) \quad (1 \leq i \leq 4), \\ &\quad \frac{1}{2}\sum_{i=1}^{4} s_i\epsilon_i \pm \frac{1}{2}(\epsilon_5 - \epsilon_7) \qquad (\prod_{i=1}^{4} s_i = -1), \\ &\quad \frac{1}{2}\sum_{i=1}^{4} s_i\epsilon_i \pm \frac{1}{2}(\epsilon_5 - \epsilon_6) \qquad (\prod_{i=1}^{4} s_i = 1), \\ &\quad \pm\epsilon_i \pm \epsilon_j \qquad (1 \leq i < j \leq 4)\}, \end{aligned}$$

where $s_i = \pm 1$. Define an order by

$$\sum_{i=1}^{7} s_i \epsilon_i > 0,$$

if $s_{\sigma(1)} = \cdots = s_{\sigma(k-1)} = 0$ and $s_{\sigma(k)} > 0$ for some $1 \leq k \leq 7$, where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \end{pmatrix}$. Then the positive roots are

$$\begin{array}{ll}
\pm \epsilon_i + \frac{1}{2}(\epsilon_6 - \epsilon_7) & (1 \leq i \leq 4), \\
\pm \frac{1}{2}\sum_{i=1}^{4} s_i \epsilon_i + \frac{1}{2}(\epsilon_5 - \epsilon_7) & (\prod_{i=1}^{4} s_i = -1), \\
\pm \frac{1}{2}\sum_{i=1}^{4} s_i \epsilon_i + \frac{1}{2}(\epsilon_5 - \epsilon_6) & (\prod_{i=1}^{4} = 1), \\
\epsilon_i \pm \epsilon_j & (1 \leq i < j \leq 4),
\end{array}$$

and simple roots are

$$\begin{aligned}
r_1 &= -\epsilon_1 + \frac{1}{2}(\epsilon_6 - \epsilon_7) = -\epsilon_1 \circ \alpha_{23} + \frac{1}{2}(\epsilon_6 - \epsilon_7) \\
r_2 &= \epsilon_3 - \epsilon_4 \\
r_3 &= \epsilon_1 - \epsilon_2 \\
r_4 &= \epsilon_2 - \epsilon_3 \\
r_5 &= \epsilon_3 + \epsilon_4 \\
r_6 &= \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) + \frac{1}{2}(\epsilon_5 - \epsilon_6) = \epsilon_1 \circ \alpha_{12} + \frac{1}{2}(\epsilon_5 - \epsilon_6).
\end{aligned}$$

Let $h_1 = h(1,0,0,0,0,0), h_2 = h(0,1,0,0,0,0)$ etc. The coroots are given by

$$H(\sum_{i=1}^{7} c_i \epsilon_i) = \sum_{i=1}^{4} c_i h_i + 2\sum_{i=5}^{7} c_i h_i,$$

where $\sum_{i=1}^{7} c_i\epsilon_i \in R$. Especially

$$
\begin{aligned}
&H(r_1) = -h_1 + (h_6 - h_7),\\
&H(r_2) = h_3 - h_4,\\
&H(r_3) = h_1 - h_2,\\
&H(r_4) = h_2 - h_3,\\
&H(r_5) = h_3 + h_4,\\
&H(r_6) = \frac{1}{2}(-h_1 - h_2 - h_3 - h_4) + (h_5 - h_6).
\end{aligned}
$$

Hence the Dynkin diagram is given by

$$
\begin{array}{ccccccccc}
r_1 & — & r_3 & — & r_4 & — & r_5 & — & r_6\\
 & & & & | & & & & \\
 & & & & r_2 & & & &
\end{array}
$$

Let

$$
\begin{aligned}
&(a)_{ij} = aE_{ij}^{(3)} \quad (1 \le i,j \le 3,\ a \in \mathfrak{C}),\\
&\mathfrak{M}_3^r = \{T \in \mathfrak{M} \text{ with real diagonal elements}\},\\
&\chi\begin{pmatrix} x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix} = x_{11} + x_{22} + x_{33}\\
&\tilde{T}(X) = \tfrac{1}{2}(TX + XT^*) \quad (X \in \mathfrak{J},\ T \in \mathfrak{M}_3).
\end{aligned}
$$

Every element of $\mathfrak{g}$ can be uniquely expressed as

$$\delta + \tilde{T},$$

where $\delta \in \mathfrak{D}_4$, $T \in \mathfrak{M}_3$ and $\chi(T) = 0$ [**F**,8.1.1]. A Chevalley system is given by

$$
\begin{aligned}
&X(\epsilon_i - \epsilon_j) = E_{i,j} - E_{4+j,4+i} && (i \neq j)\\
&X(\epsilon_i + \epsilon_j) = E_{i,4+j} - E_{j,4+i} && (i < j)\\
&X(-\epsilon_i - \epsilon_j) = E_{4+j,i} - E_{4+i,j} && (i < j)\\
&X(\epsilon_i \circ \alpha_{kl} + \frac{1}{2}(\epsilon_{4+k} - \epsilon_{4+l})) = (f_i)\widetilde{_{kl}} &&\\
&X(-\epsilon_i \circ \alpha_{kl} + \frac{1}{2}(\epsilon_{4+k} - \epsilon_{4+l})) = (f_{4+i})\widetilde{_{kl}} && (1 \leq i \leq 4,\ 1 \leq k, l \leq 3,\ k \neq l).
\end{aligned}
$$

1.8. Type E_7.

In this paragraph, we use the notations of [**H**]. Let

$$X = \{(x, y) \mid x, y \text{ are alternating } 8 \times 8 \text{ matrices}\}.$$

Define linear endomorphisms of X by

$$(x, y) \xrightarrow{p} (px + x\,{}^t\!p, -{}^t\!py - yp), \tag{1.8.1}$$

where p is an 8×8 matrix with trace 0, and

$$((x_{ij}), (y_{ij})) \xrightarrow{\vartheta} ((\sum_{m,n=1}^{8} \vartheta^{ijmn} y_{mn}), (-\sum_{m,n=1}^{8} \vartheta_{ijmn} x_{mn})), \tag{1.8.2}$$

where ϑ denotes a tensor, antisymmetric in its indices, and upper, lower indices satisfy the relation

$$\vartheta_{i_1,\ldots,i_4} = \frac{1}{4!} \sum_{j_1,\ldots,j_4} I^{1,\ldots,8}_{i_1,\ldots,i_4,j_1,\ldots,j_4} \vartheta^{j_1,\ldots,j_4}.$$

Here $I^{1,\dots,8}_{k_1,\dots,k_8}$ denotes the signature of the permutation $\binom{1,\dots,8}{k_1,\dots,k_8}$ if $\{k_1,\dots,k_8\} = \{1,\dots,8\}$, and 0 otherwise. Then, we may assume that $\mathfrak{g} = \mathfrak{E}_7$ is the linear span of these linear endomorphisms, whose Lie algebra structure is given by

$$[p,p'] = pp' - p'p, \text{ where } pp' \text{ denotes the matrix multiplication,}$$
$$[p,\vartheta] = \vartheta', \text{ where } (\vartheta')^{ijkl} = \sum_m (\vartheta^{mjkl}p_{im} + \vartheta^{imkl}p_{jm} + \vartheta^{ijml}p_{km} + \vartheta^{ijkm}p_{lm}),$$
$$[\vartheta,\vartheta'] = p, \text{ where } p_{ij} = \frac{2}{3}\sum_{l,m,n}(\vartheta^{lmni}(\vartheta')_{lmnj} - \frac{1}{8}(\sum_r \vartheta^{lmnr}(\vartheta')_{lmnr})\delta_{ij}).$$

Hereafter, we identify $p \in Lie(SL_8(\mathbf{C}))$ with the element of $\mathfrak{g}$ defined by (1.8.1). We may assume that

$$\mathfrak{h} = \{\mathrm{diag}(t_1,\dots,t_8) \mid \sum_{i=1}^{8} t_i = 0\}.$$

Let

$$\epsilon_i(\mathrm{diag}(t_i,\dots,t_8)) = t_i.$$

Then

$$R = \{\epsilon_i - \epsilon_j \quad (1 \le i,j \le 8,\ i \ne j),$$
$$\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l \quad (1 \le i < j < k < l \le 8)\}.$$

The coroots are given by

$$H(\epsilon_i - \epsilon_j) = E_i - E_j,$$
$$H(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l) = (E_i + E_j + E_k + E_l) - \frac{1}{2}\sum_{i=1}^{8} E_m.$$

Let $\vartheta(ijkl)$ be the tensor, with $(ijkl)$-coefficient $= 1$, all others zero (but to preserve the anti-symmetry of ϑ), e.g., $\vartheta(ijkl)^{ijkl} = 1$. A Chevalley system is given by

$$X(\epsilon_i - \epsilon_j) = E_{ij} \qquad (i \ne j)$$
$$X(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l) = \frac{1}{2}\vartheta(ijkl) \quad (i \le i < j < k < l \le 8).$$

In fact

$$[X(\epsilon_i - \epsilon_j), X(\epsilon_j - \epsilon_k)] = X(\epsilon_i - \epsilon_k)$$

$$[X(\epsilon_i - \epsilon_j), X(\epsilon_j + \epsilon_k + \epsilon_l + \epsilon_m)] = X(\epsilon_i + \epsilon_k + \epsilon_l + \epsilon_m)$$

$$[X(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l), X(\epsilon_i + \epsilon_m + \epsilon_n + \epsilon_r)] = X(\epsilon_i - \epsilon_s),$$

where different letters indicates different numbers. (Note that if 7 indices $(ijklmnr)$ are given, then the remaining index, say s, is uniquely determined.) The other commutators are all zero. By these commutation relations, we can show that there exists a unique involutory automorphism ι of $\mathfrak{E}_7$ such that

$$\iota(p) = -{}^t p \quad (p \in sl_8(\mathbf{C})),$$

and

$$\iota(\vartheta(ijkl)) = \vartheta(mnrs),$$

where $(mnrs)$ is chosen so that $I^{12345678}_{ijklmnrs} = 1$. Let

$$\alpha_i = \epsilon_i - \epsilon_{i+1}$$

$$\alpha_8 = \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8.$$

Then we may take as a root basis

$$\{\alpha_i \mid i \neq 1\} \quad \text{or} \quad \{\alpha_i \mid i \neq 7\}.$$

In fact, the extended Dynkin diagram is given by

$$\begin{array}{ccccccccccccc}
\alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 & — & \alpha_6 & — & \alpha_7 \\
 & & & & & & | & & & & & & \\
 & & & & & & \alpha_8 & & & & & &
\end{array}$$

The involutory automorphism ι induces the unique non-trivial automorphism of the extended Dynkin diagram.

1.9. Type E_8.

In this paragraph, we use the notations of [**VE**]. Let us consider three kinds of tensors

$$\begin{aligned} X &= (x^i_j)_{1\le i,j\le 9} \quad \text{with} \sum_{i=1}^{9} x^i_i = 0, \\ X_* &= (x_{ijk})_{1\le i,j,k\le 9}, \\ X^* &= (x^{ijk})_{1\le i,j,k\le 9}. \end{aligned}$$

Here all the tensors are assumed to be antisymmetric in the covariant indices and in the contravariant indices. We may assume that $\mathfrak{g}$ is the vector space $\{X\} \oplus \{X_*\} \oplus \{X^*\}$, which is equipped with a Lie algebra structure by

$$\begin{aligned} &[X,Y] = Z, && z^i_j = x^{\cdot}_j y^i_{\cdot} - y^{\cdot}_j x^i_{\cdot} \\ &[X,Y_*] = Z_*, && z_{ijkl} = \frac{1}{2} I^{\cdot\cdot\cdot}_{ijk} x^{\cdot}_{\cdot} y_{\cdots} \\ &[X,Y^*] = Z^*, && z^{ijk} = -\frac{1}{2} I^{ijk}_{\cdots} x^{\cdot}_{\cdot} y^{\cdots} \\ &[X^*,Y_*] = Z, && z^i_j = \frac{1}{2}(x^{i\cdot\cdot} y_{j\cdot\cdot} - \frac{1}{9} x^{\cdots} y_{\cdots} I^i_j) \\ &[X^*,Y^*] = Z_*, && z_{ijk} = \frac{1}{36} I_{ij\cdots\cdots} x^{\cdots} y^{\cdots} \\ &[X_*,Y_*] = Z^*, && z^{ijk} = \frac{1}{36} I^{ijk\cdots\cdots} x_{\cdots} y_{\cdots} \quad . \end{aligned}$$

Here we used the notations of the first two sections of [**VE**]. We may assume that $\mathfrak{h}$ is the set of the diagonal X's. Let

$$\epsilon_i(\sum_{j=1}^{9} t_i E_i) = t_i.$$

The root system is given by

$$R = \{\epsilon_i - \epsilon_j \quad (1 \le i,j \le 9,\ i \ne j), \quad \pm(\epsilon_i + \epsilon_j + \epsilon_k) \quad (1 \le i < j < k \le 9)\}.$$

The coroot are given by

$$H(\epsilon_i - \epsilon_j) = E_i - E_j,$$
$$H(\pm(\epsilon_i + \epsilon_j + \epsilon_k)) = \pm\{(E_i + E_j + E_k) - \frac{1}{3}\sum_{m=1}^{9} E_m\}.$$

Let $X_*(ijk)$ (resp. $X^*(ijk)$) be the tensor of type X_* (resp. X^*), with (ijk)-coefficient $= 1$, all others zero (but to preserve the anti-symmetry of X_* (resp. X^*)). A Chevalley system is given by

$$X(\epsilon_i - \epsilon_j) = E_{ij}$$
$$X(\epsilon_i + \epsilon_j + \epsilon_k) = X_*(ijk),$$
$$X(-\epsilon_i - \epsilon_j - \epsilon_k) = X^*(ijk).$$

Let

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \le i \le 8),$$
$$\alpha_9 = -\epsilon_1 - \epsilon_2 - \epsilon_3.$$

Then we may take as a root basis

$$\{\alpha_i \mid i \ne 8\}.$$

In fact, the extended Dynkin diagram is given by

$$\begin{array}{ccccccccccccccc}
\alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 & — & \alpha_6 & — & \alpha_7 & — & \alpha_8 \\
 & & & & | & & & & & & & & & & \\
 & & & & \alpha_9 & & & & & & & & & &
\end{array}$$

§2. Split $\mathbb{Z}$-forms.

The purpose of this section is to classify and describe the split $\mathbb{Z}$-forms of saturated, irreducible, prehomogeneous vector spaces (G, ρ, V) over $\mathbb{C}$. Here we use the definitions and the results of [G].

According to [G], first, we should choose highest weight vectors v_0 and $v_0^\vee$ of V and $V^\vee$ so that

$$V_{max}(\mathbb{Z}) \cap \mathbb{C}v_0 = V_{min}(\mathbb{Z}) \cap \mathbb{C}v_0 = \mathbb{Z}v_0,$$

where, by definition, $V_{min}(\mathbb{Z}) = \mathcal{U}_{\mathbb{Z}} \cdot v_0$ and $V_{max}(\mathbb{Z})$ is the dual lattice of $\mathcal{U}_{\mathbb{Z}} \cdot v_0^\vee$ [G]. We shall describe $V_{min}(\mathbb{Z})$ and $V_{max}(\mathbb{Z})$ explicitly for each case. Our next task is to classify the graded $\mathcal{U}_{\mathbb{Z}}$-modules $V(\mathbb{Z})$ which are $\mathbb{Z}$-lattices of V and

$$V_{min}(\mathbb{Z}) \subset V(\mathbb{Z}) \subset V_{max}(\mathbb{Z}).$$

Fortunately, it will turn out that our second task is almost nothing. In fact, our calculation will show that such a $V(\mathbb{Z})$ coincides with $V_{min}(\mathbb{Z})$ or $V_{max}(\mathbb{Z})$.

In course of our calculation, we need to fix a Chevalley system, a basis of a root system etc. In such a case, we always use those given in the first section. If a non-degenerate bilinear form $\langle , \rangle$ is defined on V, we identify the vector space $V^\vee$ with the vector space V via the isomorphism $I : V^\vee \xrightarrow{\simeq} V$ defined by $\langle v^\vee, v \rangle = \langle I(v^\vee), v \rangle$, where the left hand side is the natural pairing. (Note that I does not preserve the $\mathbb{Z}$-structure.) For the sake of a convenience for later calculations, we will give a non-degenerate bilinear form such that $\rho(G) = \rho^\vee(G)$, if we identify $V^\vee$ with V.

In (2.1)-(2.15), we shall treat reduced prehomogeneous vector spaces.

2.1. Type (1).

The representation space V can be identified with the totality of $m \times m$ matrices $M_m(\mathbb{C})$. We may assume that $G = GL_m \times GL_m$. The action of G is given by

$$\rho(g)X = g_1 X\, {}^t g_2 \quad (X \in M_m(\mathbb{C}), g = (g_1, g_2) \in G).$$

Then a highest weight vector is given by $v_0 = E_{11}$. By applying $\mathcal{U}_{\mathbb{Z}}$ to v_0, we have

$$V_{min}(\mathbb{Z}) = M_m(\mathbb{Z}).$$

We identify the dual space $V^\vee$ of V with V by $\langle X, Y\rangle = \mathrm{tr}({}^t XY)$ for $X, Y \in M_m(\mathbb{C})$. Then the action of G on $V^\vee$ is given by

$$\rho^\vee(g)Y = {}^t g_1^{-1} Y g_2^{-1} \quad (Y \in M_m(\mathbb{C}), g = (g_1, g_2) \in G).$$

Note that $\rho^\vee(G)$ is identified with $\rho(G)$ via the above identification $V = V^\vee$. A highest weight vector of $V^\vee$ is given by

$$v_0^\vee = E_{mm}.$$

Hence $V_{min}^\vee(\mathbb{Z}) = M_m(\mathbb{Z})$, and

$$V_{max}(\mathbb{Z}) = M_m(\mathbb{Z}).$$

Hence there is only one split $\mathbb{Z}$-form. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z}) = V_{max}(\mathbb{Z})$ is given by

$$E_{ij} \quad (1 \le i, j \le m)$$

and its dual is

$$E_{ij}^\vee = E_{ij} \quad (1 \le i, j \le m).$$

2.2. Type (2).

The representation space can be identified with the totality of $n \times n$ symmetric matrices $V = \{X \in M_n(\mathbb{C}) \mid {}^tX = X\}$. We may assume that $G = GL_n$. The action is given by

$$\rho(g)X = gX\,{}^tg \quad (X \in V, g \in G).$$

A highest weight vector is given by $v_0 = E_{11}$. By applying $\mathcal{U}_{\mathbb{Z}}$ to v_0, we have

$$V_{min}(\mathbb{Z}) = \{X \in M_n(\mathbb{Z}) \mid {}^tX = X\}.$$

We identify the dual space $V^\vee$ of V with V by $\langle X, Y\rangle = \mathrm{tr}\, XY$. The action of G on $V^\vee$ is given by

$$\rho^\vee(g)Y = {}^tg^{-1}Yg^{-1} \quad (Y \in V^\vee, g \in G).$$

Note that $\rho^\vee(G)$ is identified with $\rho(G)$ via the above identification $V = V^\vee$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = E_{nn}$. Since $V_{min}^\vee = \{Y \in M_n(\mathbb{Z}) \mid {}^tY = Y\}$,

$$V_{max}(\mathbb{Z}) = \sum_{i=1}^{n} \mathbb{Z}E_i + \sum_{i<j} \mathbb{Z} \cdot \tfrac{1}{2}(E_{ij} + E_{ji}).$$

We can show that $V_{max}(\mathbb{Z})/V_{min}(\mathbb{Z})$ is a simple graded $\mathcal{U}_{\mathbb{Z}}$-module. (It is enough to consider the action of the Weyl group.) Hence there are exactly two split $\mathbb{Z}$-forms. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$E_i \quad (1 \le i \le n), \quad E_{ij} + E_{ji} \quad (1 \le i < j \le n).$$

Its dual basis is given by

$$E_i^\vee = E_i \quad (1 \le i \le), \quad (E_{ij} + E_{ji})^\vee = \tfrac{1}{2}(E_{ij} + E_{ji}) \quad (1 \le i < j \le n),$$

which is a basis of $V_{max}(\mathbb{Z})$.

2.3. Type (3).

The representation space can be identified with the totality of $2m \times 2m$ skew-symmetric matrices $V = \{X \in M_{2m}(\mathbb{C}) \mid {}^tX + X = 0\}$. We may assume that $G = GL_{2m}$. The action of G is given by

$$\rho(g)X = gX\,{}^tg \quad (X \in V, g \in G).$$

A highest weight vector is given by $v_0 = E_{12} - E_{21}$. By applying $\mathcal{U}_{\mathbb{Z}}$ to v_0, we have

$$V_{min}(\mathbb{Z}) = \{X \in M_{2m}(\mathbb{Z}) \mid {}^tX + X = 0\}.$$

We identify the dual space $V^\vee$ of V with V by $\langle X, Y\rangle = -\frac{1}{2}\operatorname{tr} XY$. The action of G on $V^\vee$ is given by

$$\rho^\vee(g)Y = {}^tg^{-1}Yg^{-1} \quad (Y \in V^\vee, g \in G).$$

Note that $\rho^\vee(G)$ is identified with $\rho(G)$ via our identification. A highest weight vector of $v^\vee$ is given by $v_0^\vee = E_{2m-1,2m} - E_{2m,2m-1}$. Since $V_{min}^\vee(\mathbb{Z}) = \{Y \in M_{2m}(\mathbb{Z}) \mid Y + {}^tY = 0\}$,

$$V_{max}(\mathbb{Z}) = \{X \in M_{2m}(\mathbb{Z}) \mid X + {}^tX = 0\}.$$

Hence there is only one split $\mathbb{Z}$-form. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z}) = V_{max}(\mathbb{Z})$ is given by

$$E_{ij} - E_{ji} \quad (1 \leq i < j \leq 2m).$$

Its dual basis is

$$(E_{ij} - E_{ji})^\vee = E_{ij} - E_{ji} \quad (1 \leq i < j \leq 2m).$$

2.4. Type (4).

The representation space can be identified with the third symmetric product $S^3(\mathbb{C}^2)$ of a two dimensional vector space. We may assume that $G = GL_2 = GL(\mathbb{C}^2)$. Then G acts naturally on $S^3(\mathbb{C}^2)$. Let $e_1 = {}^t(1,0)$ and $e_2 = {}^t(0,1)$. A highest weight vector is given by $v_0 = e_1^3$. By applying $\mathcal{U}\mathbb{Z}$ to v_0, we have

$$V_{min}(\mathbb{Z}) = \mathbb{Z} \cdot e_1^3 + \mathbb{Z} \cdot 3e_1^2 e_2 + \mathbb{Z} \cdot 3e_1 e_2^2 + \mathbb{Z} \cdot e_2^3.$$

We identify the dual space $V^\vee$ of V with V itself by

$$\left\langle e_1^a e_2^{3-a}, e_1^b e_2^{3-b} \right\rangle = \begin{cases} \binom{3}{a}^{-1} & (a = b) \\ 0 & (a \neq b). \end{cases}$$

If we denote the actions of G on V and $V^\vee$ by ρ and $\rho^\vee$, respectively, then $\rho^\vee(g) = \rho({}^t g^{-1})$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_2^3$. We have

$$V_{min}^\vee(\mathbb{Z}) = \mathbb{Z} \cdot e_1^3 + \mathbb{Z} \cdot 3e_1^2 e_2 + \mathbb{Z} \cdot 3e_1 e_2^2 + \mathbb{Z} \cdot e_2^3$$

and

$$V_{max}(\mathbb{Z}) = \mathbb{Z} \cdot e_1^3 + \mathbb{Z} \cdot e_1^2 e_2 + \mathbb{Z} \cdot e_1 e_2^2 + \mathbb{Z} \cdot e_2^3.$$

We can show that $V_{max}(\mathbb{Z})/V_{min}(\mathbb{Z})$ is a simple graded $\mathcal{U}_\mathbb{Z}$-module. Hence there are exactly two split $\mathbb{Z}$-forms. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ ais given by

$$e_1^3,\ 3e_1^2 e_2,\ 3e_1 e_2^2,\ e_2^3.$$

Its dual basis is given by

$$(e_1^3)^\vee = e_1^3,\ (3e_1^2 e_2)^\vee = e_1^2 e_2,\ (3e_1 e_2^2)^\vee = e_1 e_2^2,\ (e_2^3)^\vee = e_2^3,$$

which is a basis of $V_{max}(\mathbb{Z})$.

2.5. Types (5),(6),(7),(9),(10) and (11).

Let $(l,m,n) = (3,6,1),(3,7,1),(3,8,1),(2,6,2),(2,5,3)$ or $(2,5,4)$ for the prehomogeneous vector space of type (5),(6),(7),(9),(10) or (11), respectively. Then the representation space can be identified with $V = \bigwedge^l(\mathbb{C}^m) \otimes \mathbb{C}^n$, where $\bigwedge^l(\mathbb{C}^m)$ is the l-th Grassmann product of $\mathbb{C}^m$. We may assume that $G = GL(\mathbb{C}^m) \times GL(\mathbb{C}^n)$, which acts naturally on V. Let $\{e_i \mid 1 \le i \le m\}$ and $\{f_j \mid 1 \le j \le n\}$ be the standard bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. A highest weight vector is given by $v_0 = (e_1 \wedge e_2 \wedge \cdots \wedge e_l) \otimes f_1$. By applying $\mathcal{U}_{\mathbb{Z}}$ to v_0, we have

$$V_{max}(\mathbb{Z}) = \sum_{\substack{1 \le i_1 < \cdots < i_l \le m \\ 1 \le j \le n}} \mathbb{Z} \cdot (e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j.$$

We identify the dual space $V^\vee$ of V with V by

$$\left\langle (e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j, (e_{i'_1} \wedge \cdots \wedge e_{i'_l}) \otimes f_{j'} \right\rangle = \delta_{i_1 i'_1} \cdots \delta_{i_l i'_l} \delta_{j j'},$$

where $i_1 < \cdots < i_l$, $i'_1 < \cdots < i'_l$ and δ is the Kronecker's delta. Denote the action of G on V and $V^\vee$ by ρ and $\rho^\vee$, respectively. Then $\rho^\vee(g_1, g_2) = \rho({}^t g_1^{-1}, {}^t g_2^{-1})$ for $(g_1, g_2) \in G$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = (e_{m-l+1} \wedge \cdots \wedge e_m) \otimes f_n$. Then we have $V_{min}^\vee(\mathbb{Z}) = V_{max}(\mathbb{Z})$ and $V_{max}(\mathbb{Z}) = V_{min}(\mathbb{Z})$. Hence there is exactly one split $\mathbb{Z}$-form. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z}) = V_{max}(\mathbb{Z})$ is given by

$$(e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j \quad (1 \le i_1 < \cdots < i_l \le m, 1 \le j \le n).$$

Its dual basis if given by

$$((e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j)^\vee = (e_{i_1} \wedge \cdots \wedge e_{i_l}) \otimes f_j.$$

2.6. Type (8).

The representation space can be identified with $V = S^2(\mathbb{C}^3) \otimes \mathbb{C}^2$. We may assume that $G = GL(\mathbb{C}^3) \times GL(\mathbb{C}^2)$, which acts naturally on V. Let $e_1 = {}^t(1,0,0), e_2 = {}^t(0,1,0), e_3 = {}^t(0,0,1), F_1 = {}^t(1,0)$ and $f_2 = {}^t(0,1)$. A highest weight vector of V is given by $v_0 = e_1^2 \otimes f_1$. By applying $\mathcal{U}_{\mathbb{Z}}$ to v_0, we have

$$V_{min}(\mathbb{Z}) = (\sum_{1 \leq i \leq 3} \mathbb{Z} \cdot e_i^2 + \sum_{1 \leq i < j \leq 3} \mathbb{Z} \cdot 2e_i e_j) \otimes (\mathbb{Z} f_1 + \mathbb{Z} f_2).$$

We identify the dual space $V^\vee$ of V with V by

$$\left\langle e_1^{a_1} e_2^{a_2} e_3^{a_3} \otimes f_a, e_1^{b_1} e_2^{b_2} e_3^{b_3} \otimes f_b \right\rangle = \begin{cases} \frac{a_1! a_2! a_3!}{2!}, & \text{if } (a_1, a_2, a_3, a) = (b_1, b_2, b_3, b) \\ 0, & \text{otherwise.} \end{cases}$$

Denote the actions of G on V and $V^\vee$ by ρ and $\rho^\vee$, respectively. Then $\rho^\vee(g_1, g_2) = \rho({}^t g_1^{-1}, {}^t g_2^{-1})$ $((g_1, g_2) \in G)$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_3^2 \otimes f_2$. We have $V_{min}^\vee(\mathbb{Z}) = V_{min}(\mathbb{Z})$ and

$$V_{max}(\mathbb{Z}) = \sum_{\substack{1 \leq i \leq j \leq 3 \\ 1 \leq k \leq 2}} \mathbb{Z} \cdot e_i e_j \otimes f_k.$$

We can show that $V_{max}(\mathbb{Z})/V_{min}(\mathbb{Z})$ is a simple graded $\mathcal{U}_{\mathbb{Z}}$-module. Hence there are exactly two split $\mathbb{Z}$-forms. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$\begin{aligned} &e_i^2 \otimes f_k && (1 \leq i \leq 3, 1 \leq k \leq 2), \\ &2e_i e_j \otimes f_k && (1 \leq i < j \leq 3, 1 \leq k \leq 2). \end{aligned}$$

Its dual basis is given by

$$\begin{aligned} &(e_1^2 \otimes f_k)^\vee = e_i^2 \otimes f_k && (1 \leq i \leq 3, 1 \leq k \leq 2), \\ &(2e_i e_j \otimes f_k)^\vee = e_i e_j \otimes f_k && (1 \leq i < j \leq 3, 1 \leq k \leq 2), \end{aligned}$$

which is a basis of $V_{max}(\mathbb{Z})$.

2.7. Type (12).

The representation space can be identified with $V = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$. We may assume that $G = GL(\mathbb{C}^3) \times GL(\mathbb{C}^3) \times GL(\mathbb{C}^2)$. Let $\{e_i \mid 1 \leq i \leq 3\}$ and $\{f_j \mid 1 \leq j \leq 2\}$ be the standard bases of $\mathbb{C}^3$ and $\mathbb{C}^2$, respectively. A highest weight vector is given by $v_0 = e_1 \otimes e_1 \otimes f_1$. We have

$$V_{min}(\mathbb{Z}) = \sum_{\substack{1 \leq i,j \leq 3 \\ 1 \leq k \leq k}} \mathbb{Z} \cdot e_i \otimes e_j \otimes f_k.$$

We identify $V^\vee$ with V by

$$\langle e_i \otimes e_j \otimes f_k, e_{i'} \otimes e_{j'} \otimes f_{k'} \rangle = \delta_{ii'}\delta_{jj'}\delta_{kk'}.$$

The action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2, g_3) = \rho({}^tg_1^{-1}, {}^tg_2^{-1}, {}^tg_3^{-1})$ for $(g_1, g_2, g_2) \in G$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_1 \otimes e_1 \otimes f_2$. Then we have $V_{min}^\vee(\mathbb{Z}) = V_{min}(\mathbb{Z})$ and $V_{max}(\mathbb{C}) = V_{min}(\mathbb{Z})$. Hence there is exactly one split $\mathbb{Z}$-form. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z}) = V_{max}(\mathbb{Z})$ is given by

$$e_i \otimes e_j \otimes f_k \qquad (1 \leq i, j \leq 3, 1 \leq k \leq 2).$$

Its dual basis is given by

$$(e_i \otimes e_j \otimes f_k)^\vee = e_i \otimes e_j \otimes f_k.$$

2.8. Type (13).

The representation space can be identified with $V = \mathbb{C}^{2n} \otimes \mathbb{C}^{2m}$. We may assume that $G = Sp_{2n}(\mathbb{C}) \times GL_{2m}(\mathbb{C})$. Here we realize the symplectic group $Sp_{2n}(\mathbb{C})$ as in (1.3), i.e.,

$$Sp_{2n}(\mathbb{C}) = \{g \in GL_{2n}(\mathbb{C}) \mid gJ\,{}^tg = J\}.$$

Then G acts naturally on V. Let $\{e_i \mid 1 \le i \le 2n\}$ and $\{f_j \mid 1 \le j \le 2m\}$ be the standard bases of $\mathbb{C}^{2n}$ and $\mathbb{C}^{2m}$, respectively. A highest weight vector of V is given by $v_0 = e_1 \otimes f_1$. We have

$$V_{min}(\mathbb{Z}) = \sum_{\substack{1 \le i \le 2n \\ 1 \le j \le 2m}} \mathbb{Z} \cdot e_i \otimes f_j.$$

We identify $V^\vee$ with V by the skew-symmetric bilinear form defined by

$$\begin{aligned} &\langle e_i \otimes f_j, e_{n+k} \otimes f_l \rangle = \delta_{ik}\delta_{jl}, \\ &\langle e_i \otimes f_j, e_k \otimes f_l \rangle = \langle e_{n+i} \otimes f_j, e_{n+k} \otimes f_l \rangle = 0, \end{aligned}$$

for $1 \le i, k \le n$ and $1 \le j, l \le 2m$. The action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(g_1, {}^t g_2^{-1})$ for $(g_1, g_2) \in G$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0 = e_1 \otimes f_{2m}$. Then we have $V^\vee_{min}(\mathbb{Z}) = V_{min}(\mathbb{Z})$ and $V_{max}(\mathbb{Z}) = V_{min}(\mathbb{Z})$. Hence there is exactly one split $\mathbb{Z}$-form. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z}) = V_{max}(\mathbb{Z})$ is given by

$$e_i \otimes f_j \qquad (1 \le i \le 2n,\ 1 \le j \le 2m).$$

Its dual basis is given by

$$(e_i \otimes f_j)^\vee = e_{i'} \otimes f_j,$$

where

$$i' = \begin{cases} i+n & (1 \le i \le n) \\ i-n & (n+1 \le i \le 2n). \end{cases}$$

2.9. Type (14).

Let $\{e_i\}_{1 \le i \le 6}$ be the standard basis of $\mathbb{C}^6$. The representation space can be identified with

$$V = \{ \sum_{1 \le i < j < k \le 6} x_{ijk} e_i \wedge e_j \wedge e_k \mid x_{i14} + x_{i25} + x_{i36} = 0 \quad (1 \le i \le 6)\},$$

where we regard (x_{ijk}) as an alternating tensor. We may assume that $G = \mathbb{C}^\times \times Sp_6(\mathbb{C})$, where $Sp_6(\mathbb{C})$ is realized as in (1.3). Then G acts naturally on V. A highest weight vector is given by $v_0 = e_1 \wedge e_2 \wedge e_3$. Let $1' = 4$, $2' = 5$, $3' = 6$, $4' = 1$, $5' = 2$, $6' = 3$, and $ijk = e_i \wedge e_j \wedge e_k$. Then a $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$\begin{aligned} & 123,\ 1'23,\ 12'3,\ 123',\ 12'3',\ 1'23',\ 1'2'3,\ 1'2'3', \\ (2.9.1) \qquad & 122' - 133',\ 211' - 233',\ 311' - 322', \\ & 1'22' - 1'33',\ 2'11' - 2'33',\ 3'11' - 3'22'. \end{aligned}$$

Let us define a skew-symmetric bilinear form on $\bigwedge^3(\mathbb{C}^6)$ by

$$\langle ijk, lmn \rangle = \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix},$$

where sgn is the signature on the symmetric group S_6 which is extended by

$$\operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & l & m & n \end{pmatrix} = 0, \qquad \text{if } \{ijklmn\} \neq \{123456\}.$$

Note that $X(r)$ acts on $\bigwedge^3(\mathbb{C}^6)$ as

$$(1) \qquad i \to j, \qquad j' \to -i', \qquad k \to 0 \qquad (k \neq i, j')$$

$$(2) \qquad i' \to j, \qquad j' \to i, \qquad k \to 0 \qquad (k \neq i', j')$$

or

$$(3) \qquad i \to j', \qquad j \to i', \qquad k \to 0 \qquad (k \neq i, j),$$

where $i, j \in \{1, 2, 3\}$, $1 \leq k \leq 6$, $-i = -e_i$ and $-i' = -e_{i'}$. Note also that

$$\langle ijk, i'j'k' \rangle = 1 \quad \text{and} \quad \langle ijk', i'j'k \rangle = -1$$

for $i, j, k \in \{1,2,3\}$. By using these facts, we can show that our bilinear form is $Sp_6(\mathbb{C})$-invariant. We identify V and $V^\vee$ by this bilinear form. Hence the action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(g_1^{-1}, g_2)$ for $(g_1, g_2) \in G = \mathbb{C}^\times \times Sp_6(\mathbb{C})$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = 123$. Then we have $V_{min}^\vee(\mathbb{Z}) = V_{min}(\mathbb{Z})$. The dual basis of (2.9.1) is

$$
(2.9.2) \quad
\begin{aligned}
&(123)^\vee = 1'2'3',\ (1'23)^\vee = -12'3',\ (12'3)^\vee = -1'23',\ (123')^\vee = -1'2'3,\\
&(12'3')^\vee = -1'23,\ (1'23')^\vee = -12'3,\ (1'2'3)^\vee = -123',\ (1'2'3')^\vee = 123\\
&(122' - 133')^\vee = \tfrac{1}{2}(1'2'2 - 1'3'3) \qquad \text{etc.}\\
&(1'22' - 1'33')^\vee = \tfrac{1}{2}(12'2 - 13'3) \qquad \text{etc.}
\end{aligned}
$$

Hence $V_{max}(\mathbb{Z})$ is the free $\mathbb{Z}$-module generated by (2.9.2). We can show that $V_{max}(\mathbb{Z})/V_{min}(\mathbb{Z})$ is a simple graded $\mathcal{U}_{\mathbb{Z}}$-module. Hence, there are exactly two split $\mathbb{Z}$-forms.

2.10. Type (15B).

The representation space can be identified with $V = \mathbb{C}^{2k+1} \otimes \mathbb{C}^m$. We may assume that $G = SO_{2k+1}(\mathbb{C}) \times GL_m(\mathbb{C})$. Here we realize the special orthogonal group $SO_{2k+1}(\mathbb{C})$ as in (1.2), i.e.,

$$SO_{2k+1}(\mathbb{C}) = \{g \in GL_{2k+1}(\mathbb{C}) \mid gJ\,{}^t g = J\}.$$

Then G acts naturally on V. Let $\{e_i \mid 1 \le i \le 2k+1\}$ and $\{f_j \mid 1 \le j \le m\}$ be the standard bases of $\mathbb{C}^{2k+1}$ and $\mathbb{C}^m$, respectively. A highest weight vector of V is given by $v_0 = e_1 \otimes f_1$. We have

$$V_{min}(\mathbb{Z}) = \sum_{\substack{1\le i\le 2k\\ 1\le j\le m}} \mathbb{Z}\cdot e_i \otimes f_j + \sum_{1\le j\le m} \mathbb{Z}\cdot 2e_{2k+1} \otimes f_j.$$

Let us identify V with $M_{2k+1,m}(\mathbb{C})$ by

$$\sum_{p,q} a_{pq} e_p \otimes f_q \to (a_{pq}).$$

The induced G-action on $M_{2k+1,m}(\mathbb{C})$ is given by

$$v \to g_1 v\, {}^t g_2 \qquad (g_1, g_2) \in G = SO_{2k+1} \times GL_m.$$

We identify $V^\vee$ with V by the symmetric bilinear form defined by

$$\langle v_1, v_2 \rangle = \mathrm{tr}({}^t v_1 J^{-1} v_2).$$

Then

$$\langle e_p \otimes f_q, e_r \otimes f_s \rangle = \begin{cases} 1 & (p = r' \neq 2k+1, q = s) \\ \frac{1}{2} & (p = r' = 2k+1, q = s) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$i' = \begin{cases} i + k & (1 \leq i \leq k) \\ i - k & (k+1 \leq i \leq 2k) \\ 2k+1 & (i = 2k+1). \end{cases}$$

The action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(g_1, {}^t g_2^{-1})$ for $(g_1, g_2) \in G$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_1 \otimes f_m$. Then we have $V_{min}^\vee(\mathbb{Z}) = V_{min}(\mathbb{Z})$. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$\begin{aligned} &e_i \otimes f_j && (1 \leq i \leq 2k,\ 1 \leq j \leq m), \\ &2e_{2k+1} \otimes f_j && (1 \leq j \leq m). \end{aligned}$$

Its dual basis is given by

$$(e_i \otimes f_j)^\vee = e_{i'} \otimes f_j \qquad (1 \leq i \leq 2k,\ 1 \leq j \leq m),$$
$$(2e_{2k+1} \otimes f_j)^\vee = e_{2k+1} \otimes f_j \qquad (1 \leq j \leq m).$$

Hence

$$V_{max}(\mathbb{Z}) = \sum_{\substack{1 \leq i \leq 2k+1 \\ 1 \leq j \leq m}} \mathbb{Z} \cdot e_i \otimes f_j.$$

We can show that $V_{max}(\mathbb{Z})/V_{min}(\mathbb{Z})$ is a simple graded $\mathcal{U}_{\mathbb{Z}}$-module. Hence, there are exactly two split $\mathbb{Z}$-forms.

2.11. Type (15D).

With a trivial modification of (2.10), we have

$$\langle e_p \otimes f_q, e_r \otimes f_s \rangle = \begin{cases} 1 & (p = r',\ q = s) \\ 0, & \text{otherwise,} \end{cases}$$
$$v_0 = e_1 \otimes f_1,$$
$$v_0^\vee = e_1 \otimes f_m,$$
$$V_{min}(\mathbb{Z}) = V_{max}(\mathbb{Z}) = \sum_{\substack{1 \leq i \leq 2k \\ 1 \leq j \leq m}} \mathbb{Z} \cdot e_i \otimes f_j,$$

and

$$(e_i \otimes f_j)^\vee = e_{i'} \otimes f_j.$$

In particular, there is exactly one split $\mathbb{Z}$-form.

2.12. Types (20), (21), (23) and (24).

Let $(m,n) = (2,5),\ (3,5),\ (1,6),\ (1,7)$, if we are considering a prehomogeneous vector space of type (20), (21), (23), (24), respectively. Then the representation space can be identified with $\bigwedge^{even}(\mathbb{C}^n) \otimes \mathbb{C}^m$. Here and below in this paragraph, we use the notations of (1.4). We may assume that $G = Spin_{2n} \times GL_m$, which acts naturally on V. Let $\{e_i \mid 1 \le i \le n\}$ and $\{u_j \mid 1 \le j \le m\}$ be the standard bases of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. A highest weight vector is given by $e_1 e_2 \dots e_l \otimes u_1$, where $l = 2[\frac{n}{2}]$. We have

$$V_{min}(\mathbb{Z}) = \sum_{\substack{0 \le k \le l \\ k:\text{even}}} \ \sum_{\substack{1 \le i_1 < \cdots < i_k \le l \\ 1 \le j \le m}} \mathbb{Z} \cdot e_{i_1} e_{i_2} \dots e_{i_k} \otimes u_j.$$

We identify $V^\vee$ with V by

$$\langle e_{a_i} \dots e_{a_k} \otimes u_i, e_{b_1} \dots e_{b_l} \otimes u_j \rangle = \begin{cases} 1 & (\{a_1, \dots, a_k\} = \{b_1, \dots, b_l\},\ i = j) \\ 0 & (\text{otherwise}), \end{cases}$$

where

$$1 \le a_1 < \cdots < a_k \le n,$$
$$1 \le b_1 < \cdots < b_l \le n,$$
$$1 \le i, j \le m.$$

Then the action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(\iota(g_1), {}^t g_2^{-1})$ for $(g_1, g_2) \in G = Spin_{2n} \times GL_m$. Here ι is the involutory automorphism of $Spin_{2n}$ given in (1.4). In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector is given by $1 \otimes u_m$. Then we have $V_{min}^{\vee}(\mathbb{Z}) = V_{max}(\mathbb{Z})$ and $V_{max}(\mathbb{Z}) = V_{min}(\mathbb{Z})$. Hence there is exactly one split $\mathbb{Z}$-form. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$e_{i_k} \dots e_{i_k} \otimes u_j \qquad (1 \leq i_1 < \cdots < i_k \leq n,\ k : \text{even},\ 1 \leq j \leq m),$$

and its dual basis is given by

$$(e_{i_1} \dots e_{i_k} \otimes u_j)^{\vee} = e_{i_1} \dots e_{i_k} \otimes u_j.$$

2.13. Types (27) and (28).

Let $n = 1, 2$ if we are considering a prehomogeneous vector space of type (27) or (28), respectively. The representation space can be identified with $\mathfrak{J} \otimes \mathbb{C}^n$. Here and below in this section, we use the notations of (1.6) and (1.7). We may assume that $G = G(E_6) \times GL_n$, where

$$G(E_6) = \{\text{linear automorphism of } \mathfrak{J} \text{ which preserves } det(X, Y, Z)\}.$$

See [**F**,8.1]. Then G acts naturally on V. Let $\{u_i\}$ be the standard basis of $\mathbb{C}^n$. A highest weight vector of V is given by $v_0 = E_{11}^{(3)} \otimes u_1$. We have

$$V_{min}(\mathbb{Z}) = \left(\sum_{1 \leq i \leq 3} \mathbb{Z} \cdot E_{ii}^{(3)} + \sum_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 8}} \mathbb{Z} \cdot (\tfrac{1}{2} f_j)_i \right) \otimes \sum_{1 \leq k \leq n} \mathbb{Z} \cdot u_k.$$

See (1.6) for $(a)_i$. We identify $V^{\vee}$ with V by the symmetric bilinear form defined by

$$\langle X \otimes u_j, Y \otimes u_k \rangle = \delta_{jk} \cdot \chi(X \circ Y) \qquad (X, Y \in \mathfrak{H}),$$

where χ is the trace function of $\mathfrak{J}$ (see (1.7)), and $X \circ Y = \frac{1}{2}(XY + YX)$. A direct calculation shows

$$\left\langle \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \eta_2 & y_1 \\ y_2 & \overline{y_1} & \eta_3 \end{pmatrix} \right\rangle = \sum_{i=1}^{3} \{\xi_i \eta_i + 2(x_i, y_i)\},$$

where

$$(x, y) = \tfrac{1}{2}(x\overline{y} + y\overline{x}) = \tfrac{1}{2}(\overline{x}y + \overline{y}x) \qquad (x, y \in \mathfrak{C}).$$

Let $\rho^\vee$ be the dual of ρ. Since $\chi(X \circ Y)$ is $\mathfrak{F}_4$-invariant [**F**, 4.5.13],

$$\rho^\vee(g_1, g_2) = \rho(g_1, {}^t g_2^{-1})$$

for $(g_1, g_2) \in G(F_4) \times GL_2$. Here $G(F_4)$ is the subgroup of $G(E_6)$ which corresponds to the Lie subalgebra $\mathfrak{F}_4 (\subset \mathfrak{E}_6)$ of the infinitesimal automorphisms of the Jordan algebra $\mathfrak{J}$. A direct calculation shows that

$$\chi(a_{ij}^{\sim} X \circ Y) + \chi(X \circ (-\overline{a})_{ji}^{\sim} Y) = 0 \qquad (i \neq j, a \in \mathfrak{C}, X \in \mathfrak{J}, Y \in \mathfrak{J}).$$

Hence we can define an involutory automorphism ι of $\mathfrak{E}_6$ by

$$\iota((a)_{ij}^{\sim}) = (-\overline{a})_{ji}^{\sim} \qquad (i \neq j, a \in \mathfrak{C})$$

and

$$\iota | \mathfrak{F}_4 \equiv \text{identity}.$$

Since $G(E_6) (\supset \{\omega \mid \omega^3 = 1\})$ is simply connected, ι induces an automorphism of $G(E_6)$, which we shall denote by the same letter ι. Then we have

$$\chi(gX \circ Y) = \chi(X \circ \iota(g) Y) \qquad (g \in G(E_6)).$$

Hence $\rho^\vee(g_1, g_2) = \rho(\iota(g_1), {}^t g_2^{-1})$ for $(g_1, g_2) \in G = G(E_6) \times GL_n$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = E_{33}^{(2)} \otimes u_n$. Then we have $V_{min}^\vee(\mathbb{Z}) = V_{min}(\mathbb{Z})$. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$\begin{aligned} &E_{ii}^{(2)} \otimes u_k \qquad (1 \le i \le 3, 1 \le k \le n) \\ &(\tfrac{1}{2} f_j)_i \otimes u_k \qquad (1 \le i \le 3, 1 \le j \le 8, 1 \le k \le n). \end{aligned}$$

Its dual basis is given by

$$\begin{aligned} (E_{ii}^{(3)} \otimes u_k)^\vee &= E_{ii}^{(3)} \otimes u_k \\ ((\tfrac{1}{2} f_j)_i \otimes u_k)^\vee &= -(\tfrac{1}{2} f_{\sigma(j)})_i \otimes u_k, \end{aligned}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

Hence $V_{max}(\mathbb{Z}) = V_{min}(\mathbb{Z})$. Hence there is exactly one split $\mathbb{Z}$-form.

2.14. Type (29).

In this paragraph, we use the notations of (1.8). The representation space can be identified with X. We may assume that $G = G(E_7) \times GL_1$, where $G(E_7)$ is the subgroup of $GL(X)$ which corresponds to the Lie subalgebra $\mathfrak{E}_7$ of $\mathfrak{gl}(X)$. A highest weight vector is given be $v_0 = (0, E_{18} - E_{81})$. Here and below, we choose $\{\alpha_2, \ldots \alpha_8\}$ as a basis of R. We have

$$V_{min}(\mathbb{Z}) = \sum_{1 \le i < j \le 8} (\mathbb{Z} \cdot (E_{ij} - E_{ji}, 0) + \mathbb{Z} \cdot (0, E_{ij} - E_{ji})).$$

We identify $V^\vee$ with V by the symmetric bilinear form defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = -\tfrac{1}{2}(\mathrm{tr}(x_1 y_1) + \mathrm{tr}(x_2 y_2)), \qquad (x_1, x_2), (y_1, y_2) \in X.$$

Since $G(E_7)\,(\supset \{\pm 1\})$ is simply connected, the involutory automorphism ι defined in (1.8) induces an involutory automorphism of $G(E_7)$, which we shall denote by the same letter ι. Then the action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2) = \rho(\iota(g_1), g_2^{-1})$ for $(g_1, g_2) \in G = G(E_7) \times GL_1$. In particular, $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = (E_{18} - E_{81}, 0)$. We have $V^\vee_{min}(\mathbb{Z}) = V_{min}(\mathbb{Z})$. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$(E_{ij} - E_{ji}, 0), (0, E_{ij} - E_{ji}) \qquad (1 \leq i < j \leq 8)$$

and its dual basis is given by

$$(E_{ij} - E_{ji}, 0)^\vee = (E_{ij} - E_{ji}, 0), \text{ and}$$
$$(0, E_{ij} - E_{ji})^\vee = (0, E_{ij} - E_{ji}).$$

Hence $V_{max}(\mathbb{Z}) = V_{min}(\mathbb{Z})$. Hence there is exactly one split $\mathbb{Z}$-form.

2.15. Non-regular prehomogeneous vector space with a relative invariant.

There is a unique non-regular irreducible reduced prehomogeneous vector space which has a non-trivial relative invariant, which we refer to as the type (NR) (= non-regular) provisionally in this paper. The representation space can be identified with $V = \mathbb{C}^{2n} \times S^2(\mathbb{C}^2)$. We may assume that $G = \mathbb{C}^\times \times Sp_{2n}(\mathbb{C}) \times SL_2(\mathbb{C})$, where $Sp_{2n}(\mathbb{C})$ is realized as in (1.3). (Note that $SL_2(\mathbb{C})/\{\pm\} = SO_3(\mathbb{C})$.) The first factor $\mathbb{C}^\times$ acts on V as scalar multiplications, $Sp_{2n}(\mathbb{C})$ (resp. $SL_2(\mathbb{C})$) acts naturally on $\mathbb{C}^{2n}$ (resp. $S^2(\mathbb{C}^2)$), and hence we get a G-action ρ on V. Let $\{e_i\}_{1 \leq i \leq 2n}$ (resp. $\{f_1, f_2\}$) be the standard basis of $\mathbb{C}^{2n}$ (resp. $\mathbb{C}^2$). A highest weight vector is given by $v_0 = e_1 \otimes f_1^2$. A $\mathbb{Z}$-basis of $V_{min}(\mathbb{Z})$ is given by

$$e_i \otimes f_1^2, \quad e_i \otimes 2f_1f_2, \quad e_i \otimes f_2^2, \quad (1 \leq i \leq 2n).$$

We identify $V^\vee$ with V by the skew-symmetric bilinear form on V defined by

$$
\begin{aligned}
&\langle e_i \otimes f_p f_q, e_j \otimes f_r f_s \rangle = \langle e_i, e_j \rangle \langle f_p f_q, f_r f_s \rangle ,\\
&\langle e_i, e_{n+j} \rangle = - \langle e_{n+j}, e_i \rangle = \delta_{ij},\\
&\langle e_i, e_j \rangle = \langle e_{n+i}, e_{n+j} \rangle = 0\\
&\langle f_1^2, f_1^2 \rangle = \langle f_2^2, f_2^2 \rangle = 1, \langle f_1 f_2, f_1 f_2 \rangle = \tfrac{1}{2}\\
&\langle f_p f_q, f_r f_s \rangle = 0 \quad \text{ for the other cases,}
\end{aligned}
$$

for $1 \leq i,j \leq 2n$ and $1 \leq p,q,r,s \leq 2$. Then the action $\rho^\vee$ of G on $V^\vee$ is given by $\rho^\vee(g_1, g_2, g_3) = \rho(g_1^{-1}, g_2, {}^t g_3^{-1}) \in G = \mathbb{C}^\times \times Sp_{2n}(\mathbb{C}) \times SL_2(\mathbb{C})$. In particular $\rho^\vee(G) = \rho(G)$. A highest weight vector of $V^\vee$ is given by $v_0^\vee = e_i \otimes f_2^2$. Then we have $V_{min}^\vee(\mathbb{Z}) = V_{min}(\mathbb{Z})$. A $\mathbb{Z}$-basis of $V_{max}(\mathbb{Z})$ is given by

$$
e_i \otimes f_1^2, \quad e_i \otimes f_1 f_2, \quad e_i \otimes f_2^2, \quad (1 \leq i \leq 2n).
$$

We can show that $V_{max}(\mathbb{Z})/V_{min}(\mathbb{Z})$ is a simple graded $\mathcal{U}_{\mathbb{Z}}$-module. Hence there are exactly two split $\mathbb{Z}$-forms.

2.16. Let (G_i, ρ_i, V_i) $(i = 1,2)$ be two irreducible representations and $(G_i, \rho_i^\vee, V_i^\vee)$ their duals. We assume that a Borel subgroup of each G_i is given. Let v_i and $v_i^\vee$ be highest root vectors of V_i and $V_i^\vee$, respectively. Assume that a non-degenerate bilinear form $\langle , \rangle$ is given for each V_i and that $\rho_i(G_i) = \rho_i^\vee(G_i)$, if we identify $V_i^\vee$ with V_i via this bilinear form.

Let us consider the irreducible representation $(G, \rho, V) = (G_1 \times G_2, \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ and its dual $(G, \rho^\vee, V^\vee) = (G_1 \times G_2, \rho_1^\vee \otimes \rho_2^\vee, V_1^\vee \otimes V_2^\vee)$. Highest weight vectors of $V_1 \otimes V_2$ and $V_1^\vee \otimes V_2^\vee$ are given by $v_1 \otimes v_2$ and $v_1^\vee \otimes v_2^\vee$. Then we have

$$
\begin{aligned}
V_{min}(\mathbb{Z}) &= V_{1,min}(\mathbb{Z}) \otimes V_{2,min}(\mathbb{Z}),\\
V_{max}(\mathbb{Z}) &= V_{1,max}(\mathbb{Z}) \otimes V_{2,max}(\mathbb{Z}).
\end{aligned}
$$

A non-degenerate bilinear form on V is given by

$$\langle v_1' \otimes v_2', v_1'' \otimes v_2'' \rangle = \langle v_1', v_1'' \rangle \langle v_2', v_2'' \rangle$$

for $v_i', v_i'' \in V_i$ $(i = 1, 2)$. Then $V^\vee$ can be identified with V and $\rho^\vee(G) = \rho(G)$.

Combining this fact with the calculations in (2.1)-(2.15), we have the following theorem.

2.17. Theorem. *Let (G, ρ, V) be an irreducible prehomogeneous vector space. Then there are at most two split $\mathbb{Z}$-forms which are given by $V_{max}(\mathbb{Z})$ and $V_{max}(\mathbb{Z})$. The exact number of split $\mathbb{Z}$-forms of each (G, ρ, V) is given in the following table. (The first row indicates the type of (G, ρ, V) and the second row indicates the number of split $\mathbb{Z}$-forms.)*

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
1	2	1	2	1	1	1	2	1	1	1	1	1

(14)	(15B)	(15D)	(20)	(21)	(23)	(24)	(27)	(28)	(29)	(NR)
2	2	1	1	1	1	1	1	1	1	2

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